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On the fluid limits of some loss networks

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Abstract: We study the fluid limits of loss networks under Kelly's scaling. In the case of heavy traffic for a single node and J classes of calls, we prove a degenerate diffusion approximation theorem around the corresponding fluid limit. After a careful analysis of some reflected random walks in $\mathbb{N}$, we prove that Hunt and Kurtz's conjecture (1994) for the trunk reservation policy is wrong. We conclude with some remarks on the conjecture in the case of an uncontrolled network.

Key-words: Loss networks; Diffusion approximation; Random fluid Limits

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Sur les limites fluides de certains réseaux avec perte

Résumé :

NOUS étudions les limites fluides de réseaux avec perte avec l'échelle de renormalisation de Kelly. Dans le cas d'un réseau à un nœud en forte charge, nous montrons une théorème de la limite centrale dégénéré autour de la limite fluide. En analysant de façon détaillée certaines marches aléatoires réfléchies dans $\mathbb{N}$, nous montrons que la conjecture de Hunt et Kurtz (1994) ne peut être vraie dans le cas d'une politique de service de type "trunk reservation". Nous concluons par quelques remarques sur la conjecture dans le cas où il n'y a pas de politique de contrôle.

Mots-clés : Réseaux avec perte; Théorème de la limite centrale; Limites fluides aléatoires.

1. INTRODUCTION

The loss networks we shall study in this paper are defined as follows.

- There are J links. Each link $j \leq J$ has a capacity (or a maximal number of circuits) NC_j, where $N \in \mathbb{N}$ is the scaling parameter introduced by Kelly [9]) and $C_j \in \mathbb{N}$ is the renormalized capacity of the link.
- There are R routes (or classes of customers). A route r, $1 \leq r \leq R$ occupies several links, A_{jr} circuits are required at link j, $1 \leq j \leq J$;
- The arrival process of class r customers is Poisson with parameter $N\lambda_r$. Their residence time in the network is exponential with parameter μ_r.
- The quantity $L_{N,r}(t)$ denotes the number of class r customers in the network at time t.

The free circuit process $(m_N(t)) = (m_{N,j}(t);\, j \leq J)$ is defined by, for $j \leq J$,

$$m_{N,j}(t) = NC_j - \sum_{r=1}^{R} A_{jr} L_{N,r}(t),$$

$m_{N,j}(t)$ is the number of empty places at link j at time t.

An arriving class r customer is accepted at time t if $m_N(t) \in \mathcal{A}_r$, where $\mathcal{A}_r$ is a subset of $\mathbb{N}^J$ of the form

$$\mathcal{A}_r = \{m : m_j \geq d_{jr},\, j = 1, \dots, J\} \tag{1}$$

with $(d_{jr}) \in \mathbb{N}^J$; $\mathcal{A}_r$ is the acceptance region for a call on the route r.

Hunt and Kurtz[8] have shown that the sequence of processes

$$\left(\overline{L}_N(t)\right) = \left(\frac{L_{N,r}(t)}{N};\, r \leq R\right)$$

is tight for the Skorohod topology in the space of $\mathbb{R}^R$-valued right continuous functions with left limits (provided that the sequence of the initial conditions $\left(\overline{L}_N(0)\right)$ converge). A fluid limit is one of the possible limits of this sequence.

A crucial feature of these systems shown by Hunt and Kurtz is the averaging property. The free circuit process moves much more rapidly than the process $(\overline{L}_N(t))$. It can be roughly described as follows: If at some time t,

$$\left(\overline{L}_N(t)\right) = x = (x_r;\, r \leq R),$$

at the time scale $1/N$, the process $(m_N(s))$ is a $\mathbb{N}^J$-valued Markov jump process $(\overline{m}_x(s))$ whose transitions are given by, for $m \in \mathbb{N}^J$,

$$m \to \begin{cases} m + A_r & \text{at rate } \mu_r x_r, \\ m - A_r & \text{at rate } \lambda_r 1_{\mathcal{A}_r}(m), \end{cases} \tag{2}$$

where $A_r = (A_{jr};\, j \leq J)$. Notice that the acceptance regions determine the discontinuities of the dynamic of the process. So at "time t" the future behavior of $(\overline{L}_N(t))$ is determined by the behavior at infinity of $(\overline{m}_x(s))$. If a component $j \leq J$ of $(\overline{m}_x(s))$ goes to infinity, it will imply that the link j will not play a role to reject arriving calls.

If there is a subset S of $\{1,\dots,J\}$ such that the vector $(\overline{m}_{x,j}(s);\, j \in \mathcal{S})$ converges in distribution to π^S and the components of $(\overline{m}_x(s))$ not in S converge to infinity in distribution, the infinitesimal increment of $\overline{L}_{N,r}(t)$ between t and $t+dt$ will be

$$\Big(\mu_r - \lambda_r \pi^S\left(\mathrm{proj}_S(\mathcal{A}_r)\right)\Big)\, dt,$$

where proj_S is the projection on the S-components. Under this assumption it would imply that there is a unique fluid limit (see Hunt and Kurtz [8] for more details). The probability π^S can be seen as a *local equilibrium* of the system. This lead to the conjecture.

Conjecture 1 (Hunt and Kurtz's conjecture)**.** *If $(m(t))$ is a Markov jump process on $\mathbb{N}^J$ with transition rates given by, for r, $1 \leq r \leq R$,*

$$m \to \begin{cases} m + B_r & \textit{at rate } \mu_r, \\ m - A_r & \textit{at rate } \lambda_r 1_{\{m \geq A_r\}}, \end{cases}$$

then there exist a subset S of $\{1,\dots,J\}$ and a probability distribution π^S on $\mathbb{N}^{|S|}$ such that for $x \in \mathbb{N}^S$,

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T 1_{\{m_j(u))=x_j, \forall j \in S\}}\, du = \pi^S(x) = \pi^S((x_j, j \in S)), \qquad a.s.$$

and for any $K \geq 0$,

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T 1_{\{m_j(u)\leq K, \forall j \notin S\}}\, du = 0, \qquad a.s.$$

It can be rephrased by saying that there is only one local equilibrium for this Markov jump process. This conjecture has been stated in the case where the acceptance regions are *maximal*, i.e. if there is enough room in the network, a call is accepted. A network with such acceptance rules will be called *uncontrolled network*. A similar conjecture is possible when the acceptance policy is more restrictive, for trunk reservation for example (see Section 4). Hunt [7] has shown that, for very special policies (acceptance regions are not of the form (1)), the analogue of the conjecture does not hold.

When $R = 1$ it easily shown that the conjecture is true so that there is a unique fluid limit. In Section 2 we shall be interested in the perturbation around this fluid limit (which is a J-dimensional process) in the case of an uncontrolled network. When the initial point is in the interior of the state space, the components of $(L_{N,r}(t);\, r \leq R)$ are mostly independent for a while, hence a diffusion limit theorem is easy to derive. We shall be interest when the fluid limit is on the boundary of the acceptance region and stays for a while on it (otherwise it is similar to the previous case). We show that the perturbation is a degenerated diffusion related to a $J-1$ dimensional Ornstein-Ühlenbeck process.

When $R \geq 2$, there is no general result concerning the conjecture, only partial results have been proved (see Bean *et al.* [2] and Zachary [14] for example). The main problem is to classify the possible local equilibriums, i.e. for each subset S test if π^S exists and show that the other components converges to $+\infty$ in distribution. As we shall see the last step implies that some characteristics of π^S must be precisely known. When $|S| \geq 2$ or when the sizes of some of the jumps of the Markov process are not small, it is very difficult to get some explicit information on π^S.

For the trunk reservation policy an analogous conjecture is mentioned by Hunt and Kurtz [8] and formulated by Bean *et al.* [2]. We show that it is wrong. The proof is based on precise estimates of the invariant probability of some reflected random walks in $\mathbb{N}$ (Section 3). Proposition 8 is the crucial result to get a counterexample which is worked out in Section 4. This proposition is also used to prove that in the case an uncontrolled network with two links and two routes, the conjecture is true. Despite this last result can be easily extended by adding arrivals of routes on one link, a completely general result even for the two link network (which is a simple network !) seems out of reach.

In the last section we finish by a discussion of the original conjecture. Simple examples show that the conjecture has to be reformulated, i.e. one has to restrict the set of Markov jump processes considered.

Recall that the purpose of the conjecture is of proving that, for a given initial condition, there is a unique *deterministic* fluid limit. In a different context, multi-class networks with queues, it can happen that the fluid limit is not deterministic but nevertheless the scaled processes converge to this fluid limit (see Dantzer *et al.* [4] or Dantzer and Robert [5] for some examples).

The main purpose of these rescaling procedures is to derive deterministic differential equations to get a first order picture of the process. After that step, the uniqueness question is the main problem. At some points (the discontinuities of the dynamical system), more than two solutions of the differential equations may be possible. In all the known cases of multi-class networks, this is an artifact of the procedure. It just says that some randomness remains at this point which explains this branch point for the dynamical system. With some probability, the fluid limit will follow one of these solutions. The scaling erases the diffusive randomness but not the randomness due to the discontinuities. To deal with this remaining randomness one has to go back to the small time scale (the time scale of the free circuit process in this case).

Our counterexample would lead us to believe that the conjecture is not true in general but that the convergence of the rescaled processes holds.

2. A DEGENERATE CENTRAL LIMIT THEOREM

In this section a network with one link and R routes is considered and the renormalized capacity C_1 is 1. The acceptance region is maximal: a call is accepted when there is enough room for it. It is known (see Hunt and Kurtz [8]) that, for a given initial condition, there is a unique fluid limit. The stochastic perturbation around it is analyzed and a limit theorem is proved. We consider only one case. Heavy traffic is assumed, the fluid limit starts on the boundary of the (normalized) state space and when it stays for a while on it. In the other cases it is easily shown that the components of the diffusion around the fluid limit are locally independent and related to Ornstein-Ühlenbeck processes (see Borovkov [3] for example). The main ingredient in the proof of the convergence is Proposition 2.

Notations and assumptions. Throughout this section we shall assume that for $x \geq 0$, $\mathcal{N}_x$ is a Poisson process with parameter x, $\mathcal{N}_x(dy)$ is the infinitesimal increment of the associated counting measure and $\mathcal{N}_x(]0,t])$ denotes the number of points of this process in the interval $]0,t]$. With an upper index $(\mathcal{N}_x^k)$, it denotes an i.i.d. sequence of Poisson processes with parameter x.

Heavy traffic is assumed: if $\rho_r = \lambda_r/\mu_r$, for $r \leq R$,

$$\sum_{r=1}^{R} \rho_r > 1. \tag{3}$$

This hypothesis implies that the saturation of the queue occurs with probability 1.

For $1 \leq r \leq R$ and $t \geq 0$, $L_{N,r}(t)$ is the number of class r customers at time t in the system. The initial conditions satisfy

$$\lim_{N\to+\infty} \overline{L}_{N,r}(0) = x_r, \tag{4}$$

for $r = 1, \ldots, R$ with $(x_r) \in \mathbb{R}_+^R$, $x_1 + \cdots + x_R \leq 1$.

The equations of evolution. It is easily seen that the process $(L_{N,r}(t))$ has the same distribution as the solution of the following stochastic differential equation,

$$\begin{aligned} L_{N,r}(t) = L_{N,r}(0) + \int_0^t 1_{\left\{\sum_{q=1}^R L_{N,q}(s-)<N\right\}} \mathcal{N}_{N\lambda_r}(ds) \\ - \sum_{k=1}^{+\infty} \int_0^t 1_{\{k\leq L_{N,r}(s-)\}} \mathcal{N}_{\mu_r}^k(ds). \end{aligned} \tag{5}$$

For $r = 1, \ldots, R$ the martingales associated to $(L_{N,r}(t))$ are defined by

$$\begin{aligned} M_{1,r}^N(t) &= \int_0^t 1_{\left\{\sum_{q=1}^R L_{N,q}(s-)<N\right\}} \left(\mathcal{N}_{N\lambda_r}(ds) - N\lambda_r\, ds\right), \\ M_{2,r}^N(t) &= -\sum_{k=1}^{+\infty} \int_0^t 1_{\{k\leq L_{N,r}(s-)\}} \left(\mathcal{N}_{\mu_r}^k(ds) - \mu_r\, ds\right); \end{aligned}$$

their increasing processes (see Ethier and Kurtz [6]) are given by

$$\left\langle M_{1,r}^N \right\rangle(t) = N\lambda_r \int_0^t 1_{\left\{\sum_{q=1}^R L_{N,q}(s)<N\right\}} ds, \tag{6}$$

$$\left\langle M_{2,r}^N \right\rangle(t) = \mu_r \int_0^t L_{N,r}(s)\, ds. \tag{7}$$

The equation (5) can be written as

$$(8)\quad L_{N,r}(t) = L_{N,r}(0) + M^N_{1,r}(t) + M^N_{2,r}(t) + N\lambda_r \int_0^t 1_{\left\{\sum_{q=1}^R L_{N,q}(s)<N\right\}}\,ds - \mu_r \int_0^t L_{N,r}(s)\,ds.$$

2.1. **The fluid limits.** If $(X_N(t))$ is a sequence of processes on $\mathbb{R}_+$, one defines the renormalized sequence of processes of $(X_N(t))$ by $\overline{X}_N(t) = X_N(t)/N$, for $t \geq 0$.

It is well known (See Hunt and Kurtz [8]) that the process $(\overline{L}_{N,r}(t);\, r = 1,\dots,R)$ converges in the Skorohod topology to the fluid limit $(x(t)) = (x_r(t);\, r = 1,\dots,R)$ which is the unique solution of the ordinary differential equation

$$(9)\qquad x'_r(t) = \begin{cases} \dfrac{\lambda_r}{\Lambda}\left(\langle \mu, x(t)\rangle \wedge \Lambda\right) - \mu_r x_r(t), & \text{if } \displaystyle\sum_1^R x_k(t) = 1; \\ \lambda_r - \mu_r x_r(t), & \text{if } \displaystyle\sum_1^R x_k(t) < 1; \end{cases}$$

with $x_r(0) = x_r$; $\mu = (\mu_r)$, $a \wedge b = \min(a,b)$ and $\langle \cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^R$. The dynamical system $(x_r(t);\, r = 1,\dots,R)$ lives in the region

$$\mathcal{D} = \{y \in \mathbb{R}^R_+ : y_1 + \cdots + y_R \leq 1\}.$$

It is easily seen that the condition (3) implies that for $r = 1,\dots,R$,

$$\lim_{t\to+\infty} x_r(t) = \frac{\rho_r}{\sum_1^R \rho_k};$$

therefore the stable point of $(x_r(t);\, r = 1,\dots,R)$ is $\left(\rho_r/\sum_1^R \rho_k\right)$, it lies on the boundary of $\mathcal{D}$.

We denote by

$$\Delta = \{y \in \mathcal{D} : y_1 + \cdots + y_R = 1\}, \quad \Delta^+ = \{y \in \Delta : \langle \mu, y\rangle < \Lambda\},$$

Δ^+ is the set of points of the boundary of $\mathcal{D}$ at which the dynamical system is not pushed to the interior of $\mathcal{D}$ by the dynamical system $(x_r(t))$: if $(x_r) \in \Delta^+$, the equations (9) give

$$\sum_r x'_r(0) = \sum_r \mu_r x_r - \sum_r \mu_r x_r = 0.$$

2.2. **A central limit theorem on Δ^+.** From now on we assume that the initial condition (4) is satisfied and

$$(10)\qquad \lim_{N\to+\infty} \sqrt{N}\left(\overline{L}_{N,r}(0) - x_r\right) = v_r,$$

with $(v_r) \in \mathbb{R}^R$,

$$(11)\qquad x(0) = (x_r) \in \Delta^+ \quad \text{and} \quad v_1 + \cdots + v_R = 0,$$

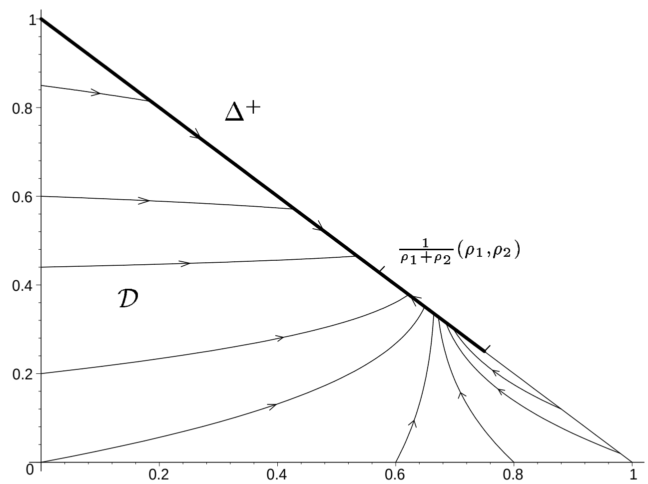

FIGURE 1. The fluid limits of a loss system with $R = 2$

i.e. the process $(L_{N,r}(t);\, r = 1, \dots, R)$ is very close to saturation at the origin when N is sufficiently large.

Since $x(0) \in \Delta^+$, it is easily seen that there exists some T such that $x(s) \in \Delta^+$ for all $s \in [0, T]$, i.e.

$$\sum_r x_r(s) = 1 \text{ and } \sup_{0\leq s\leq T} \langle \mu, x(s)\rangle < \Lambda. \tag{12}$$

Notice that in the case $R = 2$, one can take $T = +\infty$ (See Figure 1). This not the case in general, for $R = 3$ for example, but there is a region of Δ^+ containing $\left(\rho_r / \sum_1^R \rho_k\right)$ such that if $x(0)$ is in this region T is infinite (see Bean *et al.* [1]).

In the following the statements concerning the convergence in distribution of processes will refer to the Skorohod topology on the space of real valued functions on $[0, T[$ which are right continuous with left limits.

Proposition 1. *When N goes to infinity the martingale*

$$\left(\left(\sqrt{N}\, \overline{M}_{1,r}^N(t);\, r = 1, \dots, R\right), \left(\sqrt{N}\, \overline{M}_{2,r}^N(t);\, r = 1, \dots, R\right)\right)$$

converges in distribution to $(B_{1,r}(\gamma_{1,r}(t));\ 1 \leq r \leq R), B_{2,r}(\gamma_{2,r}(t));\ 1 \leq r \leq R))$, *where* $B_{1,r}$, $B_{2,r}$, $r = 1, \dots, R$ *are independent standard Brownian motions on* $\mathbb{R}$ *and for* $r = 1, \dots, R$,

$$\gamma_{1,r}(t) = \lambda_r \int_0^t \frac{\langle \mu, x(s) \rangle}{\Lambda}\, ds$$
$$\gamma_{2,r}(t) = \mu_r \int_0^t x_r(s)\, ds.$$

Proof. The increasing processes of $\left(\sqrt{N}\,\overline{M}^N_{1,r}(t)\right)$ and $\left(\sqrt{N}\,\overline{M}^N_{2,r}(t)\right)$ are given by

$$\left\langle \sqrt{N}\,\overline{M}^N_{1,r} \right\rangle (t) = \lambda_r \int_0^t 1_{\left\{\sum_{q=1}^R \overline{L}_{N,q}(s) < 1\right\}}\, ds, \tag{13}$$

$$\left\langle \sqrt{N}\,\overline{M}^N_{2,r} \right\rangle (t) = \mu_r \int_0^t \overline{L}_{N,r}(s)\, ds. \tag{14}$$

According to Hunt and Kurtz [8], when N goes to infinity the right hand side of (13) converges in distribution to

$$\lambda_r \int_0^t \frac{\langle \mu, x(s) \rangle}{\Lambda}\, ds,$$

and the right hand side of (14) to

$$\mu_r \int_0^t x_r(s)\, ds.$$

For $q, r \in \{1, \dots, R\}$, $q \neq r$ the processes

$$\left(\left\langle \overline{M}^N_{1,r}, \overline{M}^N_{2,r} \right\rangle (t)\right),\ \left(\left\langle \overline{M}^N_{1,q}, \overline{M}^N_{2,r} \right\rangle (t)\right),$$
$$\left(\left\langle \overline{M}^N_{1,q}, \overline{M}^N_{1,r} \right\rangle (t)\right) \text{ and } \left(\left\langle \overline{M}^N_{2,q}, \overline{M}^N_{2,r} \right\rangle (t)\right),$$

are identically 0 since the martingales $(M^N_{1,r}(t))$ and $(M^N_{2,r}(t))$ are stochastic integrals with respect to martingales associated with independent Poisson processes.

To conclude we apply a classical result (see Theorem 1.4 page 339 of Ethier and Kurtz [6] for example). The proposition is therefore proved. □

For $t < T$, if $Z_N(t)$ denotes the empty space in the queue at time t, i.e.

$$Z_N(t) = N - (L_{N,1}(t) + \cdots + L_{N,R}(t)),$$

from the relation (8) we get that

$$\overline{Z}_N(t) = \overline{Z}_N(0) - \sum_{r=1}^R \left(\overline{M}^N_{1,r}(t) + \overline{M}^N_{2,r}(t)\right) - \Lambda \int_0^t 1_{\{\overline{Z}_N(s) > 0\}}\, ds + \int_0^t \left\langle \mu, \overline{L}_N(s) \right\rangle\, ds. \tag{15}$$

The condition (12) implies that

$$\left(\overline{Z}_N(t);\, t \leq T\right) \xrightarrow{\text{dist.}} 0,$$

as N goes to infinity; the following proposition shows that the initial condition (11) entails a stronger statement.

Proposition 2. *The process*

$$\left(\sqrt{N}\,\overline{Z}_N(t);\, 0 \leq t < T\right) = \left(\frac{N - L_{N,1}(t) - \cdots - L_{N,R}(t)}{\sqrt{N}};\, 0 \leq t < T\right)$$

converges in distribution to 0 *as* N *goes to infinity.*

Proof. Since the limit $(x(t))$ of the process $(\overline{L}_{N,r}(t);\, r = 1, \ldots, R)$ is continuous, for $\varepsilon > 0$ and $\eta > 0$ there exists $N_0 \in \mathbb{N}$ such that for $N \geq N_0$,

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} \left|\left\langle \mu, \overline{L}_N(t)\right\rangle - \langle \mu, x(t)\rangle\right| \geq \eta\right) \leq \varepsilon;$$

the condition (12) implies that there exists $\eta > 0$ such that for $\varepsilon > 0$, there is $N_0 \in \mathbb{N}$ satisfying the following inequality: for $N \geq N_0$,

$$\text{if } \mathcal{H}_N = \left\{\sup_{0 \leq t \leq T} \left\langle \mu, \overline{L}_N(t)\right\rangle \leq \Lambda - \eta\right\} \text{ then } \mathbb{P}\left(\mathcal{H}_N\right) \geq 1 - \varepsilon. \tag{16}$$

The process $(Z_N(t))$ satisfies the stochastic integral equation

$$Z_N(t) = Z_N(0) - \sum_{r=1}^{R} \int_0^t 1_{\{Z_N(s-) > 0\}} \mathcal{N}_{\lambda_r N}(ds) + \sum_{r=1}^{R} \sum_{k=1}^{+\infty} \int_0^t 1_{\{k \leq L_{N,r}(s-)\}} \mathcal{N}^k_{\mu_r}(ds),$$

it has the same distribution as the solution of the equation (we take the same notation Z_N)

$$Z_N(t) = Z_N(0) - \sum_{r=1}^{R} \int_0^t 1_{\{Z_N(s-) > 0\}} \mathcal{N}_{\lambda_r N}(ds) + \mathcal{N}_1\left(\left]0, N\int_0^t \left\langle \mu, \overline{L}_N(s)\right\rangle\, ds\right]\right). \tag{17}$$

The process $(Z_N(t))$ can be viewed as the number of customers of an $M(t)/M/1$ queue with the service rate ΛN and $N\left\langle \mu, \overline{L}_N(t)\right\rangle$ as the instantaneous arrival rate at time t. We construct the process $(\widetilde{Z}_N(t))$ by coupling : $\widetilde{Z}_N(0) = Z_N(0)$ and

$$\widetilde{Z}_N(t) = \widetilde{Z}_N(0) - \sum_{r=1}^{R} \int_0^t 1_{\{\widetilde{Z}_N(s-) > 0\}} \mathcal{N}_{\lambda_r N}(ds) + \mathcal{N}_1\left(]0, N(\Lambda - \eta)t]\right); \tag{18}$$

the Poisson processes $\mathcal{N}_{\lambda_r N}$, $r = 1, \ldots, R$ and $\mathcal{N}_1$ in the above equation are the same as in the identity (17).

The Markov process $(X_N(t)) = (\widetilde{Z}_N(t/N))$ has the same distribution as the number of customers in an $M/M/1$ queue with arrival rate $\Lambda - \eta$ and service rate Λ. If $H(K)$

denotes the hitting time of the level K starting from 0 by this process, it is well known that if $\kappa = (\Lambda - \eta)/\Lambda$, the variable $\kappa^K H(K)$ converges in distribution to an exponentially distributed random variable with parameter η^2/Λ as K goes to infinity.

The equations (17) and (18) show that on the event $\mathcal{H}_N$ defined by (16), the inequality $Z_N(t) \leq \widetilde{Z}_N(t)$ holds for all $t \leq T$, thus for $a > 0$ and $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\leq t\leq T} \sqrt{N}\,\overline{Z}_N(t) \geq a\right) &= \mathbb{P}\left(\sup_{0\leq t\leq T} Z_N(t) \geq a\sqrt{N}\right) \\
&\leq \varepsilon + \mathbb{P}\left(\sup_{0\leq t\leq T} \widetilde{Z}_N(t) \geq a\sqrt{N}\right) \\
&= \varepsilon + \mathbb{P}_{Z_N(0)}\left(\sup_{0\leq t\leq NT} X_N(t) \geq a\sqrt{N}\right),
\end{aligned}$$

where $\mathbb{P}_b$ is the conditional probability $\mathbb{P}(\,\cdot\,|X(0) = b)$, hence using the strong Markov property of $(X(t))$ we get

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\leq t\leq T} \sqrt{N}\,\overline{Z}_N(t) \geq a\right) &\leq \varepsilon + \mathbb{P}_{Z_N(0)}\left(H\left(\lfloor a\sqrt{N}\rfloor\right) \leq NT\right) \\
&\leq \varepsilon + \mathbb{P}_{Z_N(0)}\left(H\left(\lfloor a\sqrt{N}\rfloor\right) \leq H(0)\right) + \mathbb{P}_0\left(H\left(\lfloor a\sqrt{N}\rfloor\right) \leq NT\right). \qquad (19)
\end{aligned}$$

The classical ruin probability formula (or the fact that $\left(1/\kappa^{t\wedge H(0)\wedge H\left(\lfloor a\sqrt{N}\rfloor\right)}\right)$ is a martingale) gives that

$$\mathbb{P}_{Z_N(0)}\left(H\left(\lfloor a\sqrt{N}\rfloor\right) \leq H\,(0)\right) = \frac{1/\kappa^{Z_N(0)} - 1}{1/\kappa^{\lfloor a\sqrt{N}\rfloor} - 1}$$

this term converges to 0 as N goes to infinity since $Z_N(0)/\sqrt{N} \to 0$ (see Condition (11)).

The convergence in distribution of $(\kappa^K H\,(K))$ implies that

$$\mathbb{P}_0\left(H\left(\lfloor a\sqrt{N}\rfloor\right) \leq NT\right) = \mathbb{P}_0\left(\kappa^{\lfloor a\sqrt{N}\rfloor} H\left(\lfloor a\sqrt{N}\rfloor\right) \leq N\kappa^{\lfloor a\sqrt{N}\rfloor}T\right)$$

converges to 0 as N gets large. The inequality (19) shows that the variable

$$\sup\left\{\sqrt{N}\,\overline{Z}_N(t) : 0 \leq t \leq T\right\}$$

converges in distribution to 0 as N goes to infinity. The proposition is proved. □

We define

$$\widehat{L}_{N,r}(t) = \sqrt{N}(\overline{L}_{N,r}(t) - x_r(t)) = \frac{L_{N,r}(t) - Nx_r(t)}{\sqrt{N}}.$$

According to the relation (8) the renormalized process satisfies the following identity, for $r = 1, \dots, R$ and $t \geq 0$,

$$\overline{L}_{N,r}(t) = \overline{L}_{N,r}(0) + \overline{M}^N_{1,r}(t) + \overline{M}^N_{2,r}(t) + \lambda_r \int_0^t 1_{\left\{\sum_{k=1}^R \overline{L}_{N,k}(s)<1\right\}}\,ds - \mu_r \int_0^t \overline{L}_{N,r}(s)\,ds. \tag{20}$$

The equation (9) for the fluid limits and the assumptions on the initial state gives that, for $t < T$,

$$\widehat{L}_{N,r}(t) = \widehat{L}_{N,r}(0) + \sqrt{N}\overline{M}^N_{1,r}(t) + \sqrt{N}\overline{M}^N_{2,r}(t) + \lambda_r\sqrt{N} \int_0^t 1_{\left\{\sum_{k=1}^R \overline{L}_{N,k}(s)<1\right\}}\,ds - \lambda_r\sqrt{N}\int_0^t \frac{\langle \mu, x(s)\rangle}{\Lambda}\,ds - \mu_r \int_0^t \widehat{L}_{N,r}(s)\,ds, \tag{21}$$

using the relation (15), we deduce

$$\widehat{L}_{N,r}(t) = \widehat{L}_{N,r}(0) + M^N_r(t) - \frac{\lambda_r}{\Lambda}\sqrt{N}(\overline{Z}_N(t) - \overline{Z}_N(0)) + \int_0^t \left(\frac{\lambda_r \left\langle \mu, \widehat{L}_N(s)\right\rangle}{\Lambda} - \mu_r \widehat{L}_{N,r}(s)\right) ds, \tag{22}$$

where $(M^N(t)) = (M^N_r(t))$ is the martingale defined by

$$M^N_r(t) = \sqrt{N}\left(\overline{M}^N_{1,r}(t) + \overline{M}^N_{2,r}(t)\right) - \frac{\lambda_r}{\Lambda}\sum_1^R \sqrt{N}\left(\overline{M}^N_{1,k}(t) + \overline{M}^N_{2,k}(t)\right). \tag{23}$$

Lemma 3. *The sequence of martingales $(M^N(t))$ converges in distribution to a degenerate R-dimensional Gaussian process $(G(t))$ of rank $R-1$ such that*

$$G(t) = \Gamma \cdot \Big(B_r\left(\gamma_{1,r}(t) + \gamma_{2,r}(t)\right); 1 \leq r \leq R\Big), \tag{24}$$

where $(\gamma_{1,\cdot}(t))$ and $(\gamma_{2,\cdot}(t))$ are given in Proposition 1, $(B_r(t))$ is a standard R-dimensional Brownian motion and Γ is a $R \times R$ matrix of rank $R-1$, with

$$\Gamma_{rr} = 1 - \frac{\lambda_r}{\Lambda}, \quad \Gamma_{rq} = -\frac{\lambda_r}{\Lambda},$$

for $q \neq r \in \{1, \dots, R\}$. The range of Γ is $\{y \in \mathbb{R}^R : y_1 + \cdots + y_R = 0\}$.

Proof. The convergence is a straightforward application of Proposition 1 and the continuous mapping theorem. □

Proposition 4. *Under the conditions*

$$\lim_{N\to+\infty} L_{N,r}(0)/N = x_r, \quad \lim_{N\to+\infty} (L_{N,r}(0) - Nx_r)/\sqrt{N} = v_r,$$

where the vectors $x=(x_r)$ and $v=(v_r)$ are such that $x(0)=(x_r)\in\Delta^+$ and $v_1+\cdots+v_R=0$, if $T>0$ is such that the fluid limit $(x_r(t))$ belongs to Δ^+ for $0\le t\le T$, then the vector $(\widehat{L}_{N,r}(t),\, r=0,\dots,R;\, 0\le t\le T)$ converges in distribution to the process $(X(t))$ defined by

$$X(t)=\int_0^t e^{(t-s)A}dG(s)+e^{tA}v, \tag{25}$$

where $(G(t))$ is the Gaussian process defined by (24), $v=(v_r)$ and A is the $R\times R$ matrix

$$A_{rr}=\left(\frac{\lambda_r}{\Lambda}-1\right)\mu_r \text{ and } A_{rq}=\frac{\lambda_r}{\Lambda}\mu_q,$$

for $q\neq r$.

Proof. For a process X with values in $\mathbb{R}^R$ and $\delta>0$, we denote by $w_X(\delta)$ the modulus of continuity of $(X_r(t))$ on the interval $[0,T]$,

$$w_X(\delta)=\sup_{1\le r\le R}\sup\left\{|X_r(t)-X_r(s)|:\,0\le s\le t\le T,\,t-s\le\delta\right\}.$$

We first prove that the sequence of process

$$\left(\widehat{L}_N(t)\right)=\left(\widehat{L}_{N,r}(t),\,r=0,\dots,R;\,0\le t\le T\right)$$

is tight and that any of its limiting points is continuous. For that it is sufficient to show that for any $\varepsilon,\eta>0$ there exist δ and N_0 such that the inequality

$$\mathbb{P}\left(w_{\widehat{L}_N}(\delta)\ge\varepsilon\right)<\eta$$

holds for $N\ge N_0$ (see Ethier and Kurtz [6]).

Let $t\in[0,T]$, if we set

$$G_N(t)=\sup_{1\le r\le R}\sup\left\{\left|\widehat{L}_{N,r}(s)\right|:0\le s\le t\right\},$$

the equation (22) gives the relation

$$G_N(t)\le K_N+2R\mu_*\int_0^t G_N(s)\,ds$$

with

$$K_N=\sup_{1\le r\le R}\sup_{0\le s\le T}\left(\left|\widehat{L}_{N,r}(0)\right|+\sqrt{N}\,\overline{Z}_N(s)+\left|M_r^N(s)\right|\right) \text{ and } \mu_*=\max_{1\le r\le R}\mu_r.$$

From Gronwall's inequality one gets the bound

$$G_N(t)\le K_Ne^{2R\mu_*t}, \tag{26}$$

for $0 \leq t \leq T$. Using again the equation (22) we obtain the inequality

$$\mathbb{P}\left(w_{\widehat{L}_N}(\delta) \geq \varepsilon\right) \leq \mathbb{P}\left(w_{M^N}(\delta) \geq \varepsilon/3\right) + \mathbb{P}\left(2 \sup_{0\leq t\leq T}\left(\sqrt{N}\overline{Z}_N(t)\right) \geq \varepsilon/3\right) + \mathbb{P}\left(2R\mu_* G_N(T)\delta \geq \varepsilon/3\right). \tag{27}$$

Since the sequence of processes $(M^N(t))$ converges and its limit is continuous (Lemma 3), there exist some N_0 and δ_0 such that the relation

$$\mathbb{P}\left(w_{M^N}(\delta_0) \geq \varepsilon/3\right) \leq \eta/3$$

holds for $N \geq N_0$.

From Proposition 2 we get that there exists N_1 so that if $N \geq N_1$, then

$$\mathbb{P}\left(2 \sup_{0\leq t\leq T}\left(\sqrt{N}\overline{Z}_N(t)\right) \geq \varepsilon/3\right) \leq \eta/3.$$

The initial condition (10) and the propositions 1 and 2 show that the sequence of random variables (K_N) is tight, thus there exists some constant $C_0 > 0$ such that for $N \geq 1$ the inequality $\mathbb{P}(K_N \geq C_0) \leq \eta/3$ holds.

Now if $N_2 = N_0 \vee N_1$ and

$$\delta = \delta_0 \wedge \frac{\varepsilon}{6C_0 R\mu_*} e^{-2R\mu_* T},$$

then for $N \geq N_2$, the inequality (27) gives the relation

$$\mathbb{P}\left(w_{\widehat{L}_N}(\delta) \geq \varepsilon\right) \leq \eta.$$

Thus the sequence of processes $(\widehat{L}_N(t))$ is tight and any of its limits is continuous.

If $(X(t))$ is a limit of $(\widehat{L}_N(t))$, using the relation (22) we get that $(X(t))$ satisfies the stochastic integral equation

$$X_r(t) = v_r + G_r(t) + \int_0^t \sum_{q=1}^{R} A_{rq} X_q(s)\, ds, \quad r = 1, \ldots, R. \tag{28}$$

The range of A is $\mathcal{S} = \{y \in \mathbb{R}^R : y_1 + \cdots + y_R = 0\}$, since $(G(t))$ is a Gaussian process in $\mathcal{S}$, the above equation can be rewritten as a classical non degenerate linear stochastic differential equation in $\mathcal{S}$. In particular there is a unique strong solution (see Ethier and Kurtz [6] or Rogers and Williams [13]). It is easy to check that the process $(X(t))$ defined by (25) verifies (28). The proposition is proved. □

3. Analysis of some reflected random walks in $\mathbb{N}$

In this section we consider a Markovian jump process $(X(t))$ on $\mathbb{N}$ having only two possible jump lengths c_r, $r = 1, 2$, it is assumed that $0 < c_1 < c_2$. The rates are given by, for

$r = 1, 2$,

$$m \to \begin{cases} m + c_r & \text{at rate } \mu_r; \\ m - c_r & \text{at rate } \lambda_r 1_{\{m \geq c_r\}}, \end{cases}$$

with $\lambda_r > 0$ and $\mu_r > 0$. We assume that c_1 and c_2 are relatively prime, i.e. $\gcd(c_1, c_2) = 1$ so that the jump process is irreducible on $\mathbb{N}$ and such that

$$\sum_r c_r(\lambda_r - \mu_r) > 0 \tag{29}$$

so that it is ergodic with a unique invariant distribution denoted by ν.

In the following an explicit expression of the invariant distribution ν is given and then used to discuss the sign of the quantities, for $r = 1, 2$,

$$\lambda_r \nu([c_r, +\infty[) - \mu_r.$$

As we shall see in Section 4, this discussion is crucial to determine the behavior of the fluid limits of some networks with two links. The above Markovian jump process has been already analyzed in Bean *et al.* [1], but our study requires more precise expressions for their invariant probability.

The invariant distribution ν is the unique solution of the system of global balance equations:

$$\left(\sum_r \mu_r + \sum_r \lambda_r 1_{\{m \geq c_r\}} \right) \nu(m) = \sum_r \lambda_r \nu(m + c_r) + \sum_r \mu_r 1_{\{m \geq c_r\}} \nu(m - c_r). \tag{30}$$

For $m \geq c_2$ (recall that $c_2 > c_1$) it has the recursive expression

$$\begin{aligned} (\mu_1 + \mu_2 + \lambda_1 + \lambda_2)\, \nu(m) = \lambda_2 \nu(m + c_2) + \lambda_1 \nu(m + c_1) \\ + \mu_1 \nu(m - c_1) + \mu_2 \nu(m - c_2). \end{aligned} \tag{31}$$

The probability ν is related to the polynomial P defined by

$$\begin{aligned} P(x) &= \lambda_2 x^{2c_2} + \lambda_1 x^{c_2 + c_1} - (\mu_1 + \mu_2 + \lambda_1 + \lambda_2) x^{c_2} + \mu_1 x^{c_2 - c_1} + \mu_2 \\ &= (\lambda_2 x^{c_2} - \mu_2)(x^{c_2} - 1) + (\lambda_1 x^{c_1} - \mu_1) x^{c_2 - c_1} (x^{c_1} - 1). \end{aligned} \tag{32}$$

According to Bean *et al.* [1] the polynomial P has exactly one root, p_1 in the interval $]0, 1[$ and, among the other roots, excepting $x = 1$, $c_2 - 1$ roots p_i, $2 \leq i \leq c_2$ have a modulus strictly less than p_1. The other $c_2 - 1$ roots have a modulus strictly greater than 1.

The expression of ν is given by, for $m \in \mathbb{N}$,

$$\nu(m) = \sum_1^{c_2} \alpha_i (1 - p_i) p_i^m, \tag{33}$$

where $\alpha_1, \ldots, \alpha_{c_2}$ are complex coefficients; they are determined by the following proposition.

Proposition 5. *If the roots p_i, $2 \leq i \leq c_2$, of the polynomial P are all different, then for $2 \leq i \leq c_2$,*

$$\alpha_i = -\alpha_1 \frac{(1-p_1)^2}{(1-p_i)^2} \frac{(1-p_i^{c_1})p_i^{c_2-c_1}}{(1-p_1^{c_1})p_1^{c_2-c_1}} \frac{\lambda_2 p_1^{c_2} - \mu_2}{\lambda_2 p_i^{c_2} - \mu_2} \prod_{\substack{2 \leq j \leq c_2 \\ j \neq i}} \frac{p_1 - p_j}{p_i - p_j}$$

with the normalization relation $\alpha_1 + \cdots + \alpha_{c_2} = 1$.

Proof. The coefficients $\alpha_1, \ldots, \alpha_{c_2}$ are obtained by using the equilibrium equation (30) for $m = 0, \ldots, c_2 - 2$. If the expression (33) is used for these values, we get

$$\sum_{i=1}^{c_2} \alpha_i (1-p_i) p_i^m (\lambda_2 p_i^{c_2} + \lambda_1 p_i^{c_1} - \mu_1 - \mu_2) = 0$$

for $m = 0, \ldots, c_1 - 1$ and

$$\sum_{i=1}^{c_2} \alpha_i (1-p_i) p_i^{m-c_2} (\lambda_2 p_i^{2c_2} + \lambda_1 p_i^{c_2+c_1} - (\lambda_1 + \mu_1 + \mu_2) p_i^{c_2} + \lambda_1 p_i^{c_2-c_1} = 0$$

for $m = c_1, \ldots, c_2 - 2$.

Since p_i, $1 \leq i \leq c_2$, is a root of P, the representations (32) of the polynomial P show that the vector $(\alpha_1, \ldots, \alpha_{c_2})$ is the unique solution of the following linear system,

(34) $$\alpha_1 + \alpha_2 + \cdots + \alpha_{c_2} = 1,$$

(35) $$\sum_{i=1}^{c_2} \alpha_i (1-p_i) p_i^m \left((1-p_i^{c_1}) p_i^{c_2-c_1}\right)^{-1} (1-p_i^{c_2-c_1})(\lambda_2 p_i^{c_2} - \mu_2) = 0$$

for $0 \leq m \leq c_1 - 1$ and

(36) $$\sum_{i=1}^{c_2} \alpha_i (1-p_i) p_i^{m-c_2} (\lambda_2 p_i^{c_2} - \mu_2) = 0$$

for $c_1 \leq m \leq c_2 - 2$; the solution is unique because the invariant probability is unique.

The equations (35) and (36) form a linear system with respect to $\alpha_2, \ldots, \alpha_{c_2}$; these coefficients can be thus expressed as a function of α_1. The associated matrix is denoted by $\mathcal{R}$.

Definition 6. *The square matrix $\mathcal{R}$ of size $c_2 - 1$ is defined by*

$$\mathcal{R} = (R_{m,i};\ 0 \leq i, m \leq c_2 - 2)$$

with

$$R_{m,i-2} = (1-p_i) p_i^m \left((1-p_i^{c_1}) p_i^{c_2-c_1}\right)^{-1} (1-p_i^{c_2-c_1})(\lambda_2 p_i^{c_2} - \mu_2), \qquad 0 \leq m \leq c_1 - 1,$$

$$R_{m,i-2} = (1-p_i) p_i^{m-c_2} (\lambda_2 p_i^{c_2} - \mu_2), \qquad c_1 \leq m \leq c_2 - 2,$$

for $2 \leq i \leq c_2$; $\mathcal{R}_i$ is the matrix obtained from $\mathcal{R}$ by replacing p_i by p_1.

The fact that $\mathcal{R}$ is indeed non singular and the computation of its determinant $\det(\mathcal{R})$ are given in the appendix (see Lemma 15). The expression of the α_i, for $2 \leq i \leq c_2$ as functions of α_1 is then obtained using Cramer's formulas

$$\alpha_i = -\alpha_1 \frac{\det(\mathcal{R}_i)}{\det(\mathcal{R})}.$$

The proposition is proved. □

The following technical lemma is proved in the appendix.

Lemma 7. *If $\mathcal{E}$ is the set of parameters $(\lambda_1, \mu_1, \lambda_2, \mu_2) \in \mathbb{R}_+^4$ such that the polynomial*

$$P(x) = \lambda_2 x^{2c_2} + \lambda_1 x^{c_2+c_1} - (\mu_1 + \mu_2 + \lambda_1 + \lambda_2)x^{c_2} + \mu_1 x^{c_2-c_1} + \mu_2$$

has a root of multiplicity at least two, the Lebesgue measure of $\mathcal{E}$ is 0.

We can now prove the main result of this section.

Proposition 8. *For $r = 1, 2$, if $r' = 3 - r$, the quantities*

$$\lambda_r \nu([c_r, +\infty[) - \mu_r, \quad \lambda_r p_1^{c_r} - \mu_r \text{ and } \left(\frac{\mu_{r'}}{\lambda_{r'}}\right)^{1/c_{r'}} - \left(\frac{\mu_r}{\lambda_r}\right)^{1/c_r},$$

have the same sign (with the convention that if one of them is 0, the two others are also 0).

Proof. We first assume that the roots p_i, $2 \leq i \leq c_2$, of the polynomial P are all distinct. Since p_1 is a root of the polynomial P defined by (32), we have

$$(\lambda_2 p_1^{c_2} - \mu_2)(1 - p_1^{c_2}) + (\lambda_1 p_1^{c_1} - \mu_1) p_1^{c_2-c_1}(1 - p_1^{c_1}) = 0;$$

the relation $0 < p_1 < 1$ implies that $\lambda_2 p_1^{c_2} - \mu_2$ and $\lambda_1 p_1^{c_1} - \mu_1$ have opposite signs. Thus, for $r = 1, 2$, the quantities $\lambda_r p_1^{c_r} - \mu_r$ and $(\mu_{r'}/\lambda_{r'})^{(1/r')} - (\mu_r/\lambda_r)^{(1/r)}$ have the same sign. Therefore, we just have to prove that $\lambda_r \nu([c_r, +\infty[) - \mu_r$ and $\lambda_r p_1^{c_r} - \mu_r$ have the same sign.

At equilibrium the mean drift of the jump process is 0, therefore the following relation holds,

$$\sum_r c_r (\lambda_r \nu([c_r, +\infty[) - \mu_r) = 0. \tag{37}$$

Consequently, the quantities $\lambda_r \nu([c_r, +\infty[) - \mu_r$, $r = 1, 2$ have opposite signs.

The representation (33) of the invariant probability and the normalization relation give that

$$\lambda_2 \nu([c_2, +\infty[) - \mu_2 = \sum_{i=1}^{c_2} \alpha_i (\lambda_2 p_i^{c_2} - \mu_2).$$

According to the proof of Proposition 5, $\alpha_i = -\alpha_1 \det(\mathcal{R}_i)/\det(\mathcal{R})$ for $2 \leq i \leq c_2$, thus one gets the identity

$$\lambda_2 \nu([c_2, +\infty[) - \mu_2 = \alpha_1 \frac{\det(\mathcal{S})}{\det(\mathcal{R})}, \tag{38}$$

where $\mathcal{S}$ is the $c_2 \times c_2$ matrix defined as follows.

Definition 9. *The $c_2 \times c_2$ matrix $\mathcal{S}$ is defined by*

$$\mathcal{S} = (S_{m,i};\ 0 \le i, m \le c_2 - 1),$$

where, for $i = 1, \dots, c_2$, $S_{m,i-1}$ is defined as

$$\begin{cases} \lambda_2 p_i^{c_2} - \mu_2, & m = 0 \\ (1-p_i)p_i^{m-1}\left((1-p_i^{c_1})p_i^{c_2-c_1}\right)^{-1}(1-p_i^{c_2-c_1})(\lambda_2 p_i^{c_2} - \mu_2), & 1 \le m \le c_1 \\ (1-p_i)p_i^{m-1-c_2}(\lambda_2 p_i^{c_2} - \mu_2), & c_1 + 1 \le m \le c_2 - 1. \end{cases}$$

Using the expression of $\det(\mathcal{S})$ and $\det(\mathcal{R})$ given in Proposition 15 and 16, the relation (38) implies

$$\lambda_2 \nu([c_2, +\infty[) - \mu_2 = c_1 \alpha_1 \frac{(1-p_1)(\lambda_2 p_1^{c_2} - \mu_2)}{(1-p_1^{c_1})p_1^{c_2-c_1}} \prod_{2 \le i \le c_2} \frac{p_1 - p_i}{1 - p_i}. \tag{39}$$

It is easily seen that the product in the last expression is necessarily non negative. The constant α_1 is positive (see Bean *et al.* [1]). From the identity (39) we conclude that the quantities

$$\lambda_2 \nu([c_2, +\infty[) - \mu_2 \text{ and } \lambda_2 p_1^{c_2} - \mu_2$$

have the same sign.

If the roots p_i, $2 \le i \le c_2$ are not necessarily distinct, Lemma 7 shows that the set $\mathbb{R}^4 - \mathcal{E}$ is dense in $\mathbb{R}^4$. Thus there exists a sequence (ω_n) such that

$$\omega_n = (\lambda_{n,1}, \mu_{n,1}, \lambda_{n,2}, \mu_{n,2}) \notin \mathcal{E}$$

for all n and $\lim_n \omega_n = (\lambda_1, \mu_1, \lambda_2, \mu_2)$. We can assume that for all $n \in \mathbb{N}$

- ω_n satisfies the relation (29). The corresponding invariant probability distribution is denoted by ν_n and $p_{n,i}$, $i = 1, \dots, c_2$ are the real roots in $[0,1[$ of the polynomial P_n defined by (32) associated with the parameters of ω_n. By numbering properly the roots of the polynomial P_n, for $i = 1, \dots, c_2$, the convergence $\lim_n p_{n,i} = p_i$ holds.
- $\lambda_{n,2} p_{n,1}^{c_2} - \mu_{n,2}$ is not 0 for all $n \in \mathbb{N}$, otherwise if the terms of an infinite subsequence are null the result is clear from the identity (39).

A straightforward application of Theorem 3.3 of Malyshev and Men'shikov [12] show that the invariant probabilities ν_n converge weakly to the invariant distribution ν. According to the relation (39)

$$\frac{\lambda_{n,2}\nu_n([c_2, +\infty[) - \mu_{n,2}}{(\lambda_{n,2} p_{n,1}^{c_2} - \mu_{n,2})} = c_1 \alpha_{n,1} \frac{(1-p_{n,1})}{(1-p_{n,1}^{c_1})p_{n,1}^{c_2-c_1}} \prod_{2 \le i \le c_2} \frac{p_{n,1} - p_{n,i}}{1 - p_{n,i}}$$

Therefore, to prove that $\lambda_2 \nu([c_2, +\infty[) - \mu_2$ and $\lambda_2 p_1^{c_2} - \mu_2$ have the same sign, it is sufficient to show that $\limsup_n \alpha_{n,1} > 0$.

The linear system of equations (34), (35) and (36) associated with the parameters of ω_n converge to the linear system associated with $(\lambda_1, \mu_1, \lambda_2, \mu_2)$. Since each of these linear systems has a unique solution, we conclude that when n goes to infinity the solutions

$(\alpha_{n,i},\, 1 \leq i \leq c_2)$ converge to $(\alpha_i,\, 1 \leq i \leq c_2)$. In particular $\lim_n \alpha_{n,1} = \alpha_1$, and this last quantity is positive. The proposition is proved. □

4. THE EXISTENCE OF RANDOM FLUID LIMITS

In this section we now consider a two-link network with two routes $r = 1, 2$, with

$$A_1 = (a_1, b_1) = (1,1) \text{ and } A_2 = (a_2, b_2) = (2,1),$$

the route $r = 2$ requires 2 free circuits on link 1 and one on link 2. The route $r = 1$ needs one free circuit on each link. As before λ_r [resp. μ_r] denotes the arrival [resp. departure] rate of calls on route r. The renormalized capacities of the links 1 and 2 are respectively defined by $C_1 = 3$ and $C_2 = 2$. In this setting $X_0 = (1,1)$ is on the boundary of the renormalized state space, i.e. (see the relations (41) below) are satisfied. Recall that $\mathcal{A}_r$ is the domain of acceptance of a type r customer in the state space $(\mathbb{N} \cup \{\infty\})^2$ of the free circuits of the network.

More precisely the process $(\overline{m}_x(t))$ defined by the relation (2) is, in this case, a Markov process on $(\mathbb{N} \cup \{\infty\})^2$ with transitions given by, for $r = 1, 2$,

$$m \to \begin{cases} m + (a_r, b_r) & \text{at rate } \mu_r x_r \\ m - (a_r, b_r) & \text{at rate } \lambda_r 1_{\mathcal{A}_r}(m). \end{cases} \tag{40}$$

In order to study the local equilibriums defined in the introduction, we shall consider the cases $S = \{1\}$ and $S = \{2\}$. We shall analyze the free circuit process when one of the two links is near saturation and the other link has a very large number of free circuits. The two corresponding Markov processes on $\mathbb{N}$ have the following transitions:
Case $S = \{1\}$, a process $(\overline{m}^1_x(t))$ with transitions defined by , for $r = 1, 2$,

$$n \to \begin{cases} n + a_r & \text{at rate } \mu_r x_r \\ n - a_r & \text{at rate } \lambda_r 1_{\mathcal{A}_r}((n, \infty)). \end{cases}$$

Case $S = \{2\}$, a process $(\overline{m}^2_x(t))$ with transitions defined by , for $r = 1, 2$,

$$n \to \begin{cases} n + b_r & \text{at rate } \mu_r x_r \\ n - b_r & \text{at rate } \lambda_r 1_{\mathcal{A}_r}((\infty, n)). \end{cases}$$

These two processes are the projections of $(\overline{m}_x(t))$ on each coordinate axis when the other coordinate is set to be infinite.

It is always assumed that for $j = 1, 2$, the process $(\overline{m}^j_x(t))$ is irreducible. This implies in particular that $\{a_r,\ a_r \neq 0\}$ and $\{b_r,\ b_r \neq 0\}$ are not empty and each of them has a gcd 1 and that (a_1, a_2) and (b_1, b_2) are not proportional. When it exists the invariant probability of $(\overline{m}^j_x(t))$ is denoted by π^j_x.

The following subset $\mathbb{W}^+$ of $\mathbb{R}^2_+$ has been introduced by Bean *et al.* [2] (see also Zachary [14]). If x is in $\mathbb{W}^+$, then there is a local equilibrium for $S = \{1\}$ and $S = \{2\}$.

Definition 10. *The set* $\mathbb{W}^+$ *is the subset of* $[0, C_1] \times [0, C_2]$ *of elements* (x_1, x_2) *satisfying the following conditions:*

– *(x_1, x_2) is on the boundary of the (fluid) state space (the two links are saturated),*

$$a_1x_1 + a_2x_2 = C_1, \quad b_1x_1 + b_2x_2 = C_2 \tag{41}$$

– *The fluid limit stays locally on this boundary*

$$\begin{cases} a_1(\lambda_1 - \mu_1 x_1) + a_2(\lambda_2 - \mu_2 x_2) > 0 \\ b_1(\lambda_1 - \mu_1 x_1) + b_2(\lambda_2 - \mu_2 x_2) > 0 \end{cases} \tag{42}$$

– *At the microscopic level, the Markov jump process on $\mathbb{N}^2$ of free circuits can diverge along the two directions,*

$$b_1(\lambda_1 \pi_x^1(\mathcal{A}_1) - \mu_1 x_1) + b_2(\lambda_2 \pi_x^1(\mathcal{A}_2) - \mu_2 x_2) \leq 0, \tag{43}$$

$$a_1(\lambda_1 \pi_x^2(\mathcal{A}_1) - \mu_1 x_1) + a_2(\lambda_2 \pi_x^2(\mathcal{A}_2) - \mu_2 x_2) \leq 0, \tag{44}$$

with the convention that if $r \in \{1, 2\}$, $x \in \mathbb{R}_+^2$ then $\pi_x^r(F) = \pi_x^r(proj_{\{r\}}(F))$ for $F \subset \mathbb{N}$, where $proj_{\{r\}}$ is the projection on the rth coordinate.

If $\mathbb{W}^+$ is non empty, it implies that the free circuit process

$$(\overline{m}_x(t)) = (\overline{m}_x^1(t), \overline{m}_x^2(t))$$

is not ergodic, and that it can diverge along one of the two axes. In this case the process $(\overline{m}_x(t))$ converges in distribution to

$$p\,\delta_\infty \otimes \pi^2 + (1-p)\,\pi^1 \otimes \delta_\infty,$$

where $0 < p < 1$ and π^1 and π^2 are probability distributions on $\mathbb{N}$. With probability p [resp. $1-p$] the first [resp. second] component converges to $+\infty$. In other words the associated fluid limits of this network can be indeed random. Zachary [14] has conjectured the set $\mathbb{W}^+$ is empty with the acceptance regions defined by (1) considered here. We first show that this conjecture does not hold. This conjecture that $\mathbb{W}^+$ is non empty is related to Hunt and Kurtz's conjecture for a two-link network.

Notice that for general random walks on $\mathbb{N}^2$ this situation is not uncommon if the transitions on the axes are properly set (see Malyshev and Men'shikov [12] for example). This is not the case for the free circuit processes: the transitions (40) on the axes are the same as in the interior provided that the process remains in $\mathbb{N}^2$.

4.1. **A counterexample with trunk reservation.** Trunk reservation is a control mechanism to give priorities and to prevent instabilities. The acceptance regions are subsets of $(\mathbb{N} \cup \{+\infty\})^2$ defined by

$$\mathcal{A}_1 = \{m : m_1 \geq 1 \text{ and } m_2 \geq 1\},$$
$$\mathcal{A}_2 = \{m : m_1 \geq 2 \text{ and } m_2 \geq a\},$$

where $a \in \mathbb{N}$, $a > 1$. A route of type 1 is accepted if there is enough place and a route of type 2 requires a free circuits on link 2 instead of 1.

The associated free circuit processes corresponding to the point $X_0 = (1,1)$ are thus defined as follows. The Markov process $(\overline{m}^1(t))$ on link 1 (we omit the subscript X_0 of $(\overline{m}^1_{X_0}(t))$) has the transitions

$$m \to \begin{cases} m+1 & \text{at rate } \mu_1 \\ m-1 & \text{at rate } \lambda_1 1_{\{m\geq 1\}} \\ m+2 & \text{at rate } \mu_2 \\ m-2 & \text{at rate } \lambda_2 1_{\{m\geq 2\}}. \end{cases} \tag{45}$$

Similarly the transitions of the Markov process $(\overline{m}^2(t))$ are given by

$$m \to \begin{cases} m+1 & \text{at rate } \mu_1+\mu_2 \\ m-1 & \text{at rate } \lambda_1 1_{\{m\geq 1\}} + \lambda_2 1_{\{m\geq a\}}. \end{cases} \tag{46}$$

As before, for $i = 1, 2$, π^i denotes the invariant distribution of $(\overline{m}^i(t))$ when it exists.

Proposition 11. *There exist parameters λ_1, μ_1, λ_2, $\mu_2 \in \mathbb{R}_+$ for a network with two links and a trunk reservation parameter a such that the set $\mathbb{W}^+$ is non empty.*

Proof. We show that the element X_0 can be an element of $\mathbb{W}^+$. This will be the case if the following set of inequalities is satisfied,

$$\begin{cases} \lambda_1 - \mu_1 + 2(\lambda_2 - \mu_2) & > 0 \\ \lambda_1 - \mu_1 + \lambda_2 - \mu_2 & > 0, \end{cases} \tag{47}$$

and

$$\lambda_1\pi^2([1,+\infty[) - \mu_1 + 2(\lambda_2\pi^2([a,+\infty[) - \mu_2) \leq 0 \tag{48}$$

$$\lambda_1\pi^1([1,+\infty[) - \mu_1 + \lambda_2\pi^2([2,+\infty[) - \mu_2 \leq 0. \tag{49}$$

We assume that the relations (47) hold. It clearly implies the existence of the invariant probabilities π^1 and π^2.

Since at equilibrium the average drift is 0 for the Markov processes $(\overline{m}^1(t))$ and $(\overline{m}^2(t))$, the identities

$$\lambda_1\pi^1([1,+\infty[) - \mu_1 + 2(\lambda_2\pi^1([2,+\infty[) - \mu_2) = 0$$
$$\lambda_1\pi^2([1,+\infty[) - \mu_1 + 2(\lambda_2\pi^2([a,+\infty[) - \mu_2) = 0$$

hold, thus the inequalities (48) and (49) are equivalent to the relations

$$\lambda_1\pi^2([1,+\infty[) - \mu_1 \geq 0 \tag{50}$$

$$\lambda_1\pi^1([1,+\infty[) - \mu_1 \leq 0. \tag{51}$$

Since the results of section 3 apply to the process $(\overline{m}^1(t))$, Proposition 8 shows that the inequality (51) is implied by the condition

$$\mu_1/\lambda_1 \geq (\mu_2/\lambda_2)^{1/2}.$$

The invariant probability π^2 is given by

$$\pi^2(m) = \begin{cases} \pi^2(0)\left(\dfrac{\mu_1+\mu_2}{\lambda_1}\right)^m & \text{for } m = 0,\dots,a-1 \\ \pi^2(0)\left(\dfrac{\mu_1+\mu_2}{\lambda_1}\right)^{a-1}\left(\dfrac{\mu_1+\mu_2}{\lambda_1+\lambda_2}\right)^{m-a+1} & \text{for } m \geq a, \end{cases}$$

where the renormalizing constant $\pi^2(0)$ is given by

$$\pi^2(0) = \left(1 - \frac{\mu_1+\mu_2}{\lambda_1}\right) \Big/ \left(1 - \frac{\lambda_2}{\lambda_1+\lambda_2-\mu_1-\mu_2}\left(\frac{\mu_1+\mu_2}{\lambda_1}\right)^a\right),$$

if $\lambda_1 \neq \mu_1+\mu_2$. If $\mu_1+\mu_2 < \lambda_1$, then $\pi^2(0)$ converges to $1-(\mu_1+\mu_2)/\lambda_1 \leq 1-\mu_1/\lambda_1$ when a tends to $+\infty$, therefore there exists a_0 such that the relation (50) is satisfied for $a \geq a_0$.

If the parameters λ_1, μ_1, λ_1, μ_1 and a are such that $a \geq a_0$ and

$$\mu_1+\mu_2 < \lambda_1, \quad (\mu_2/\lambda_2)^{1/2} \leq \mu_1/\lambda_1, \text{ and } 2\lambda_2 > \mu_2,$$

the inequalities (47), (48) and (49) hold; hence $X_0 \in \mathbb{W}^+$. The proposition is proved. □

The problem of the emptiness of $\mathbb{W}^+$ remains open in case of uncontrolled access to the network, i.e. without trunk reservation. We give a partial answer in this case.

4.2. **The case of uncontrolled access.** The acceptance regions are maximal, i.e.

$$\mathcal{A}_r = \{m \in (\mathbb{N}\cup\{\infty\})^2, m_1 \geq a_r \text{ and } m_2 \geq b_r\},$$

for $r = 1, 2$. (A call is accepted if there is room for it).

Proposition 12. *For a two-link network with two types of routes and maximal acceptance regions, the set $\mathbb{W}^+$ is empty.*

Proof. If x is an element of $\mathbb{W}^+$, it satisfies the relations (42) (43 and (44); the last two inequalities can be rewritten in this case as

$$\sum_r a_r(\lambda_r\pi_x^2([b_r,+\infty[) - \mu_r x_r) \leq 0 \tag{52}$$

$$\sum_r b_r(\lambda_r\pi_x^1([a_r,+\infty[) - \mu_r x_r) \leq 0. \tag{53}$$

Assume first that $a_1a_2b_1b_2 \neq 0$. As before, since the mean drift is 0 for the probability π_x^1, we have

$$\lambda_2\pi_x^1([a_2,+\infty[) - \mu_2x_2 = -\frac{a_1}{a_2}(\lambda_1\pi_x^1([a_1,+\infty[) - \mu_1x_1),$$

hence the inequality (53) is equivalent to

$$\left(b_1 - b_2\frac{a_1}{a_2}\right)\left(\lambda_1\pi_x^1([a_1,+\infty[) - \mu_1x_1\right) \leq 0.$$

The same argument for π_x^2 and the inequality (52) give the condition

$$\left(a_1 - a_2\frac{b_1}{b_2}\right)\left(\lambda_1\pi_x^2([b_1,+\infty[) - \mu_1x_1\right) \leq 0.$$

The irreducibility of the process $(m_x(t))$ implies the relation $a_1b_2 - b_1a_2 \neq 0$.

If $a_1b_2 - b_1a_2 > 0$ then, according to Proposition 8, the two previous inequalities are equivalent to the relations

$$(\mu_2 x_2/\lambda_2)^{1/a_2} \geq (\mu_1 x_1/\lambda_1)^{1/a_1}$$

and

$$(\mu_1 x_1/\lambda_1)^{1/a_1} \geq (\mu_2 x_2/\lambda_2)^{1/a_2}.$$

In particular the inequalities

$$(\mu_1 x_1/\lambda_1)^{a_2/a_1} \leq \mu_2 x_2/\lambda_2 \leq (\mu_1 x_1/\lambda_1)^{b_2/b_1}$$

would hold, and therefore, since $a_2/a_1 < b_2/b_1$, we would get $\mu_1 x_1/\lambda_1 \geq 1$ and $\mu_2 x_2/\lambda_2 \geq 1$. Therefore the inequalities (42) cannot be satisfied.

The case $a_1b_2 - b_1a_2 < 0$ is similar.

The case where one of $a_1,\ a_2,\ b_1$ and b_2 is zero is easy. Let us assume that, for example, $a_1 = 0$. Using again the fact that the mean drift is 0 at equilibrium (Equation (37)) in the relations (52) and (53), we obtain

$$\sum_{r=1,2} a_r\lambda_r \left(\pi_x^2([b_r,+\infty[) - \pi_x^1([a_r,+\infty[)\right) \leq 0,$$
$$\sum_{r=1,2} b_r\lambda_r \left(\pi_x^1([a_r,+\infty[) - \pi_x^2([b_r,+\infty[)\right) \leq 0.$$

Since $a_2 \neq 0$ (by irreducibility of $(m_x(t))$, the first inequality gives the relation

$$\pi_x^2([b_2,+\infty[) - \pi_x^1([a_2,+\infty[) \leq 0. \tag{54}$$

and from the second inequality one gets that

$$b_1\lambda_1 \left(1 - \pi_x^2([b_1,+\infty[)\right) + b_2\lambda_2 \left(\pi_x^1([a_2,+\infty[) - \pi_x^2([b_2,+\infty[)\right) \leq 0.$$

Since $b_1 \geq 1$, by ergodicity $\pi_x^2([b_1,+\infty[) < 1$, the relation

$$b_2\lambda_2 \left(\pi_x^1([a_2,+\infty[) - \pi_x^2([b_2,+\infty[)\right) < 0,$$

holds, contradicting the inequality (54). The set $\mathbb{W}^+$ is therefore empty. The proposition is proved. □

5. Some remarks on the original conjecture

In this section we give some simple example where the conjecture 1 does not hold. As we shall see, this example do not correspond to a loss network. This implies that the conjecture has to be reformulated. At the same occasion it shows that the behavior of Markov processes whose transitions are given by the relations

$$m \to \begin{cases} m + B_r & \text{at rate } \mu_r \\ m - A_r & \text{at rate } \lambda_r 1_{\{m \geq A_r\}}, \end{cases} \tag{55}$$

can be fairly complicated.

A simple null recurrent model. The transitions of the Markov process $(X(t)) = (X_1(t), X_2(t))$ on $\mathbb{N}^2$ are defined by

$$(x,y) \to \begin{cases} (x+1,y) & \text{at rate } \mu \\ (x,y+1) & \text{at rate } \mu \\ (x-1,y-1) & \text{at rate } \lambda 1_{\{x\geq 1,\, y\geq 1\}}. \end{cases} \tag{56}$$

If we set $B_1 = (1,0)$, $B_2 = (0,1)$, $\mu_1 = \mu_2 = \mu$, $A_1 = A_2 = 0$ and $A_3 = (1,1)$, $\lambda_3 = \lambda$ and $B_3 = 0$, then this model is in the class of Markov processes defined in (55).

This process is a queueing model with two queues, the arrivals at each queue are independent and their rate is μ, service occurs simultaneously in both queues at rate λ if there is at least one customer in each queue.

Proposition 13. *If $\mu < \lambda$ then $(X(t))$ is null recurrent and for any $\varepsilon > 0$ and $C > 0$ there exists $K > 0$ such that the relations*

$$\begin{cases} \lim_{T\to+\infty} \dfrac{1}{T}\displaystyle\int_0^T 1_{\{X_1(s)\wedge X_2(s)\leq K\}}\, ds \geq 1-\varepsilon, \\ \lim_{T\to+\infty} \dfrac{1}{T}\displaystyle\int_0^T 1_{\{X_1(s)\vee X_2(s)\geq C\}}\, ds = 1 \end{cases} \tag{57}$$

hold almost surely, with $a \wedge b = \min(a,b)$ and $a \vee b = \max(a,b)$.

Proof. We set

$$(Z(t)) = (Z_1(t), Z_2(t)) = \Big(X_1(t)\wedge X_2(t),\; X_1(t) - X_2(t)\Big)\,;$$

$(Z(t))$ is also a Markov process in $\mathbb{N}^2$. The component $(Z_2(t))$ is a symmetrical birth and death process on $\mathbb{Z}$ with jump rate μ, therefore $(Z(t))$ cannot be ergodic. It is easily seen that $(Z_1(t))$ can be stochastically bounded by the number of customers in a $M/M/1$ queue with arrival rate μ and service rate λ (with the same method as in the proof of Proposition 2). Hence by ergodicity of this queue, for $\varepsilon > 0$, there exists some K such that, almost surely,

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T 1_{\{Z_1(s)\leq K\}}\, ds \geq 1-\varepsilon.$$

If (τ_n) is the sequence of hitting times of 0 by $(Z_1(t))$ then by the strong Markov property, the sequence $(Z_2(\tau_n))$ behaves as a (discrete) symmetrical random walk on $\mathbb{Z}$. The process $(Z(t))$ is thus recurrent, hence null recurrent. A classical result for null recurrent processes (see Kemeny *et al.* [10] for example) is that they spend a negligible amount of time within finite sets. Thus, for any constant C, almost surely,

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T 1_{\{X_1(s)\vee X_2(s)\geq C\}}\, ds = 1.$$

The proposition is proved. □

Corollary 14. *If $\mu < \lambda$, the Markov process defined by (56) does not satisfy Conjecture 1.*

Proof. If the conjecture is true, there exists a subset S of $\{1, 2\}$ and a probability distribution π on $\mathbb{N}^{|S|}$ such that, for $x \in \mathbb{N}^{|S|}$,

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathbb{P}(X_j(s) = x_j,\, \forall j \in S)\, ds = \pi((x_j,\, j \in S)) \qquad a.s. \tag{58}$$

and for any $K \geq 0$,

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathbb{P}\left(X_j(u) \leq K,\, \forall j \notin S\right)\, du = 0 \qquad a.s. \tag{59}$$

Null recurrence implies that either the relation (59) or the relation (58) is not satisfied if S is the empty set or $\{1, 2\}$.

Therefore $S = \{1\}$ or $S = \{2\}$. If $X_1(0) = X_2(0) = 0$, then by symmetry $\mathbb{P}(X_1(t) \leq K) = \mathbb{P}(X_2(t) \leq K)$ for all $t \geq 0$, in particular

$$\liminf_{T\to+\infty} \frac{1}{T}\int_0^T \mathbb{P}(X_1(s) \leq K)\, ds = \liminf_{T\to+\infty} \frac{1}{T}\int_0^T \mathbb{P}(X_2(s) \leq K)\, ds;$$

if the conjecture is true one of the members of the equation must be 0 and the other positive. Contradiction. The corollary is proved. $\square$

A sequence of similar examples can be constructed as follows: For $N \in \mathbb{N}$, a Markov process in $\mathbb{N}^N$ whose transitions are given by, for $m \in \mathbb{N}^N$ and i, $1 \leq i \leq N$,

$$m \to \begin{cases} m + e_i & \text{at rate } \mu \\ m - e_i - e_{i+1} & \text{at rate } \lambda \end{cases}$$

with e_i the ith unit vector of $\mathbb{N}^N$ and by convention $i + 1$ is understood modulo N. This process is clearly transient whenever $\mu > 2\lambda$. We believe that if $\mu < 2\lambda$, the process is ergodic if N is odd and null recurrent otherwise. When N is even, the process diverges in two directions: either the odd coordinates grow or the even coordinates.

The common feature of these Markov processes with the counterexample seen for trunk reservation is that the process can diverge along the two axes. The Markov processes for the Hunt and Kurtz's conjecture must be such that if one of the terms $\lambda_r A_r$ or $\mu_r B_r$ is positive, the other one must be also positive.

Appendix

Proposition 15. *Under the assumption of Proposition 5 the matrix $\mathcal{R}$ of Definition 6 is non singular and its determinant is given by*

$$\det(\mathcal{R}) = K \prod_{2\leq i<j\leq c_2} (p_j - p_i) \prod_{2\leq i\leq c_2} \frac{(1-p_i)^2(\lambda_2 p_i^{c_2} - \mu_2)}{(1-p_i^{c_1})p_i^{c_2-c_1}}, \tag{60}$$

where K is given by

$$K = (-1)^{c_1(c_2-c_1+1)} \prod_{i=2}^{c_2} \frac{1-\omega_i^{c_1}}{1-\omega_i},$$

with $\omega_i = \exp(2(i-1)\pi/c_2)$ *for* $i = 1, \dots, c_2$.

Proof. After factorization, this determinant can be written as

$$\det(\mathcal{R}) = \prod_{i=2}^{c_2} \frac{(1-p_i)(\lambda_2 p_i^{c_2} - \mu_2)}{(1-p_i^{c_1})p_i^{c_2-c_1}}\ Q(p_2, \dots, p_{c_2}),$$

where $Q(X_2, \dots, X_{c_2})$ is the determinant of the matrix

$$\begin{pmatrix} 1 - X_2^{c_2-c_1} & \dots & 1 - X_i^{c_2-c_1} & \dots & 1 - X_{c_2}^{c_2-c_1} \\ (1 - X_2^{c_2-c_1})X_2 & \dots & (1 - X_i^{c_2-c_1})X_i & \dots & (1 - X_{c_2}^{c_2-c_1})X_{c_2} \\ \dots & \dots & \dots & \dots & \dots \\ (1 - X_2^{c_2-c_1})X_2^{c_1-1} & \dots & (1 - X_i^{c_2-c_1})X_i^{c_1-1} & \dots & (1 - X_{c_2}^{c_2-c_1})X_{c_2}^{c_1-1} \\ 1 - X_2^{c_1} & \dots & 1 - X_i^{c_1} & \dots & 1 - X_{c_2}^{c_1} \\ (1 - X_2^{c_1})X_2 & \dots & (1 - X_i^{c_1})X_i & \dots & (1 - X_{c_2}^{c_1})X_{c_2} \\ \dots & \dots & \dots & \dots & \dots \\ (1 - X_2^{c_1})X_2^{c_2-c_1-2} & \dots & (1 - X_i^{c_1})X_i^{c_2-c_1-2} & \dots & (1 - X_{c_2}^{c_1})X_{c_2}^{c_2-c_1-2} \end{pmatrix}.$$

We note that Q is a polynomial in the variables $X_2, \dots, X_{c_2}$, of degree at most $c_2 - 1$ in each variable, it is zero when two variables are equal or when one of them is 1. Hence,

$$Q(X_2, \dots, X_{c_2}) = K V_{c_2-1}(X_2, \dots, X_{c_2}) \prod_{i=2}^{c_2} (1 - X_i) \tag{61}$$

where the Vandermonde determinant of order n is denoted by

$$V_n(X_1, \dots, X_n) = \prod_{1 \le i < j \le n} (X_j - X_i)$$

and K is a constant which depends only on c_1 and c_2.

Let us prove that K is not zero by proving that $Q(\omega_2, \dots, \omega_{c_2})$ is not zero where $\omega_2, \dots, \omega_{c_2}$ are the complex roots of $X^{c_2} - 1$ different from 1.

Simple algebra gives

$$Q(\omega_2, \dots, \omega_{c_2}) = (-1)^{c_1(c_2-c_1)} \delta \prod_{i=2}^{c_2} (1 - \omega_i^{c_1}),$$

where δ is the determinant of the matrix

$$\begin{pmatrix} 1 & \dots & 1 & \dots & 1 \\ \omega_2 & \dots & \omega_i & \dots & \omega_{c_2} \\ \dots & \dots & \dots & \dots & \dots \\ \omega_2^{c_2-c_1-2} & \dots & \omega_i^{c_2-c_1-2} & \dots & \omega_{c_2}^{c_2-c_1-2} \\ \omega_2^{c_2-c_1} & \dots & \omega_i^{c_2-c_1} & \dots & \omega_{c_2}^{c_2-c_1} \\ \dots & \dots & \dots & \dots & \dots \\ \omega_2^{c_2-1} & \dots & \omega_i^{c_2-1} & \dots & \omega_{c_2}^{c_2-1} \end{pmatrix}.$$

As for the Vandermonde determinant, it is easily seen that

$$\delta = \sigma_{c_1,c_2}(\omega_2, \dots, \omega_{c_2}) V_{c_2-1}(\omega_2, \dots, \omega_{c_2})$$

where σ_{c_1,c_2} is the elementary symmetric polynomial of degree c_1 in $c_2 - 1$ variables, i.e.

$$\sigma_{c_1,c_2}(X_1, \dots, X_{c_2-1}) = \sum_{1 \le i_1 < \dots < i_{c_1} \le c_2-1} X_{i_1} \cdots X_{i_{c_1}}.$$

Since $\omega_2, \dots, \omega_{c_2}$ are the roots of $X^{c_2-1} + \dots + X + 1$, the quantity

$$(-1)^{c_1} \sigma_{c_1,c_2}(\omega_2, \dots, \omega_{c_2})$$

is the coefficient of $X^{c_2-c_1-1}$ in this polynomial. Hence $\sigma_{c_1,c_2}(\omega_2, \dots, \omega_{c_2}) = (-1)^{c_1}$ and therefore δ is not zero.

As c_1 and c_2 are relatively prime, the product $\prod_{i=2}^{c_2}(1-\omega_i^{c_1})$ is not zero, hence $Q(\omega_2, \dots, \omega_{c_2}) \neq 0$, consequently $K \neq 0$.

The expression (60) for $\det(\mathcal{R})$ has been established. To conclude we have to prove that this determinant is not 0. Since the p_i's are different, all we have to show is that $\lambda_2 p_i^{c_2} - \mu_2 \neq 0$ for $i = 2,\dots,c_2$. Otherwise if there is an $i > 2$ such that $p_i^{c_2} = \lambda_2/\mu_2$, then, using the expression (32) of the polynomial P, we get that $p_i^{c_1} = \lambda_1/\mu_1$. Thus one would have $|p_i|^{c_2} = \lambda_2/\mu_2$ and $|p_i|^{c_1} = \lambda_1/\mu_1$. Using again (32), it implies that $|p_i|$ is a root of P in the interval $]0, 1[$. Contradiction, since p_1 is the only root of P in $]0, 1[$. The proposition is proved. □

Proposition 16. *The determinant of the matrix $\mathcal{S}$ of definition 9 is given by*

$$(-1)^{c_2} c_1 K \prod_{1 \le i < j \le c_2} (p_j - p_i) \prod_{1 \le i \le c_2} \frac{(1-p_i)(\lambda_2 p_i^{c_2} - \mu_2)}{(1-p_i^{c_1}) p_i^{c_2-c_1}},$$

where K is the same constant as in Proposition 15.

Proof. The quantity $\det(\mathcal{S})$ is computed with the same arguments as for $\det(\mathcal{R})$. The factorization gives the identity

$$\det(\mathcal{S}) = \prod_{i=1}^{c_2} \frac{(1-p_i)(\lambda_2 p_i^{c_2} - \mu_2)}{(1-p_i^{c_1}) p_i^{c_2-c_1}} Q_1(p_1, \dots, p_{c_2})$$

where Q_1 is a polynomial with c_2 variables and it is easy to see that

$$Q_1(X_1, \dots, X_{c_2}) = K_1 V_{c_2}(X_1, \dots, X_{c_2})$$

for some constant K_1. With the same notations as in the previous proof, we get that

$$Q_1(\omega_1, \dots, \omega_{c_2}) = c_1 Q(\omega_2, \dots, \omega_{c_2})$$

where Q is defined by (61), consequently $K_1 = (-1)^{c_2} c_1 K$. The proposition is proved. □

We conclude with the proof that the polynomial P defined by (32) does not have multiple roots for almost every $(\lambda_1, \mu_1, \lambda_2, \mu_2)$, with respect to Lebesgue measure on $\mathbb{R}_+^4$.

Proof of Lemma 7. Without any loss of generality, we can assume that the parameters are such that $\mu_1 + \mu_2 + \lambda_1 + \lambda_2 = 1$, hence

$$P(x) = \lambda_2 x^{2c_2} + \lambda_1 x^{c_2+c_1} - x^{c_2} + \mu_1 x^{c_2-c_1} + \mu_2.$$

The polynomial P has a root of multiplicity at least 2 if and only if its discriminant $D(\lambda_1, \mu_1, \lambda_2, \mu_2)$ is 0 (see Lang [11]). This discriminant is a multivariate polynomial with respect to $(\lambda_1, \mu_1, \lambda_2, \mu_2)$. Consequently, either it is identically 0, or the set of its zeroes is Lebesgue negligible. It is easily seen that the coefficient of highest degree with respect to μ_2 is not identically 0. The lemma is proved. □

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